Problem 7

For the following series, write formulas for the sequences a_n, S_n, and R_n, and find the limits of the sequences as $n \to \infty$ (if the limits exist).

$$\frac{3}{1 \cdot 2} - \frac{5}{2 \cdot 3} + \frac{7}{3 \cdot 4} - \frac{9}{4 \cdot 5} + \cdots$$

Solution

$$a_n = (-1)^{n+1} \frac{2n+1}{n(n+1)} = (-1)^{n+1} \left(\frac{1}{n} + \frac{1}{n+1} \right)$$

$$\begin{aligned} S_n &= \sum_{i=1}^{n} (-1)^{i+1} \left(\frac{1}{i} + \frac{1}{i+1} \right) \\ &= \left(\frac{1}{1} + \frac{1}{2} \right) - \left(\frac{1}{2} + \frac{1}{3} \right) + \cdots + (-1)^{n+1} \left(\frac{1}{n} + \frac{1}{n+1} \right) \\ &= 1 + \frac{(-1)^{n+1}}{n+1} \end{aligned}$$

$$S = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \left[1 + \frac{(-1)^{n+1}}{n+1} \right] = 1$$

$$R_n = S - S_n = 1 - \left[1 + \frac{(-1)^{n+1}}{n+1} \right] = \frac{(-1)^{n+2}}{n+1} = \frac{(-1)^n}{n+1}$$

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} (-1)^{n+1} \left(\frac{1}{n} + \frac{1}{n+1} \right) = \left[\lim_{n \to \infty} (-1)^{n+1} \right] (0 + 0) = 0$$

$$\lim_{n \to \infty} R_n = \lim_{n \to \infty} \frac{(-1)^n}{n+1} = 0$$